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The Uses Of Mathesis

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The word "uses," in the title of this Article, is intended only in its broader significations. We shall not touch, unless incidentally, upon the obvious applications of the mathematics to the practical arts of life, the arts of measurement and manufacture; but shall speak of their value in the cultivation of the intellectual, moral, and spiritual faculties; of their service in leading the student to a higher appreciation of the whole sphere of human joys, a clearer understanding of all objects of human thought, a better performance of all human duties.

The word "mathematics," on the other hand, we shall use in the older and narrower sense; not meaning, as Peirce has defined them, the science which draws necessary conclusions; but, for the present, restricting the word to signify the sciences of space, of number, and of time. This is not from any dissatisfaction with the wider definition, which is also true; but simply because we have no other word by which to class together geometry, algebra, and arithmetic; and it is these

of which we wish particularly to speak. They are three entirely discrete sciences, and were originally classed together because they were the only sciences in which deductive reasoning could be used; the advance of thought has in our century brought various forms of mechanics under the same category, and very properly included them under the same name. Nevertheless there are other points in which the sciences of space, time, and number deserve to be considered apart; and it is to them that the present Article is devoted.

The relation of space and time to the Divine Author of the universe is a matter beyond the reach of the human understanding, and a matter of no special consequence to our present considerations. Whether space and time be merely forms of human thought, having nothing directly answering to them in nature, and in the Creator of nature; or be modes of the divine existence, attributes of God's eternal substance; or be entities self-existent and independent of the Almighty Being and his power; in either case, the majority of our conclusions will hold good, and we shall, we trust, make manifest to the patient reader, the great value of geometry, algebra, and arithmetic as a foundation for all aesthetic, moral, and religious culture. But the reader who holds different views of the nature of space and time must not be surprised to find us betraying our faith in the common view of the unlearned, neither holding with Spinoza, nor with Carlyle's Kant, nor with Auguste Comte.

Geometry gives us the science of space in its formal or logical aspect; dealing with magnitudes of one, two, or three dimensions, and with zeros in one, two, or three dimensions, also with directions and with imaginary motions. Algebra deals with time in its flow, past or future, with zeros, or instants, and with quantity in one dimension. The language of algebra is then applied to space, and gives birth to the calculus, in its various forms from Descartes to Laugier; and more recently to quaternions and stigmatic geometry, which succeed in giving geometrical interpretations to all the ordinary forms of algebraic language. Arithmetic has, for

its material, only the abstract relation of number, which is suggested neither by space nor time in themselves; since in space there is no division of parts or aspects; and in time there is no division, except that made by ourselves in conscious thought; the simplest being into past and future. The relations of number are, then, suggested only by the material creation, and it is from the visible and tangible that we get our idea of number to apply to space and time.

All the properties of number, therefore, and all the operations of algebra, may be illustrated by forms in space; and for this reason the titles “geometry “and “geometer” are frequently used, in a larger sense, to include mathematics and mathematicians of every grade.

The modes in which geometry may be pursued are, at least, four. In its origin, as its name implies, it was cultivated simply as a practical art, for practical ends. Afterward it became a potent stimulant to fancy and imagination. Thirdly, it was made the first of demonstrative sciences, and remains to the present day the very type and pattern of deductive reasoning. Finally, it is the basis and root of the fine arts of statuary and drawing, and thus of painting. The practical uses of the mathematics, in the ordinary sense of the word practical, we pass by. In the wider sense of the word practical, we might say that what is useful is practical, and that the culture of the imagination, the reason, and the higher tastes, — inasmuch as it conduces most powerfully to human enjoyment and usefulness, — is the most practical of all ends.

The use of geometry in cultivating the fancy and imagination has, in modern days, been too much overlooked. Arithmetic, probably from its obvious utility, has been allowed by far too large a place in the common schools of this country, and geometry has been crowded out. This course has resulted in giving less quickness and expertness even in figures to the student, wearied of the over-work. The first lessons which nature gives in the school of life, are lessons in form. The child has been in the world but a few hours, when it begins to trace the outlines of the window-sash against the sky. In

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a few months it has learned to recognize a multitude of objects by their shape, while as yet its notions of time and of number are wholly undeveloped. It learns to see likenesses and understand pictures, drawings of the shapes of things, of hundreds of things, before it begins to count, or to estimate the lapse of time. Careful experiments on children have convinced us that in the ordinary mode of neglecting the study of geometry, and forcing the arithmetical power, there is an actual diminution of the power to see the truths of space, and to imagine forms, during the first six or eight years of schooling. The course of nature evidently would be to train from the beginning this geometrical power, which the child displays as the first evidence of intellectual life. The imagination of form, or shape, being the first act of, imagination, is also the most valuable among the earlier acts. It is capable by training of being led to the utmost precision, and such training is the first lesson in accuracy of thought. The differences between the integral numbers is so great that the arithmetic of whole numbers gives but little training in exactness. In the approximation by decimal fractions, the pupil is taught to consider anything beyond the third or fifth place as too small to be considered; and thus, although learning, if properly taught, the great importance of knowing to how many places he should carry his approximation so as to get sufficient precision and waste no labor in superfluous nicety, he gets no training in the conception of absolute accuracy; that comes in the algebraic notation, by which he represents his fraction as a vulgar fraction, or his square root as a surd; not in decimal notation. It comes also more forcibly, and in vastly more interesting form, in his geometrical definitions and imaginations. More than once we have taken a boy of ten or twelve years old, and in fifteen minutes conversation drawn out of him—without telling him anything, but simply leading him by questions to see for himself, and tell us — a complete description of the genesis of an ellipse and ellipsoids of revolution, with the form and peculiarities of their sections by a plane. We began by a goat tethered to a

ring sliding on a cord fastened at each end to a stake; then passed to a dove tethered to a ring sliding on a thread fastened at each end to a telegraph pole; then for the rings put points; and for thread and cord, lines; and the thing was done. The child had passed from the concrete in material things to the concrete in pure space; from the approximation of a ring and cord to the absolute precision of a point and line; accurate not only to the fifth place of decimals, but to innumerable infinitudes of places — absolutely accurate. It may have been his first lesson in the clear, definite, precise exercise of the imagination. One such lesson may be a matter of great value to the intellectual life; a course of such lessons must be of benefit.

Definiteness, clearness of imagination, is one of the most important of intellectual qualities; and it is very apt to be wanting. It is important, because we cannot even observe correctly without it. No object is visible at a single view; the attention goes from part to part, and it requires imagination or memory to integrate the parts. Without clear and precise power of imagination the whole impression is thus confused, and the observer cannot tell what he has observed. The proof that this failing is common may be found by asking a man to draw with his pencil, from memory, a sketch of the most familiar and the most simple outline, and seeing how wide he will come of the mark, and how utterly unable he will be to correct his own drawing; or it may be found by appealing to the lawyers, who will assure you of the impossibility of getting any ordinary witness to describe accurately any object or any event which he is known to have seen. The reader may, indeed, test himself, and see how distinctly he can recall the form and appearance of the most familiar object not now in his sight, whether a piece of furniture, the face and figure of a dear friend, a favorite tree, or favorite landscape, and he will probably be astonished to find how faint and indistinct his recollection of some important details will prove to be.

Yet the whole work and business of life depends very

largely upon the accuracy of memory, and upon the vividness of imagination. The imagination has a double function, of recollection and of invention; in one man it performs the one better, and in another the other. But, in one or both of its forms, its action is absolutely essential to the correct performance of every duty of life, and the enjoyment of every pleasure, — to the understanding of every problem. The present has no duration; it is but the zero-point dividing the past from the future, and our lives consist in our memories, or our anticipations; the zero of the present betrays, by the conversion of our purposes into our history, the clearness and accuracy of our imagination, and the strength of our will; it shows what use we are making of memory and invention in guiding us to this conversion of our future into our past.

Geometry is the first study which directly and largely cultivates the power of imagination. The forms of matter suggest to us *a priori* laws of space, and stimulate us to the invention of new laws. The material object does not conform perfectly to the law which it suggests; even in the minutest crystal the microscope would probably betray some slight variation from a perfect figure; in the larger crystals the naked eye readily detects the variation. But the conformity is sufficient to suggest, irresistibly, the perfect form; and when, under the stimulus of this suggestion, we have invented new *a priori* laws, we frequently find that nature has anticipated us, although we knew it not, and been building for ages upon the law which we have just invented. So constantly has this been the case, that Whewell, in his Philosophy of the Inductive Sciences, makes it the normal method of the discovery of scientific laws; he thinks that we always invent them *a priori,* as hypotheses, and then find them embodied in the universe as facts.

But it is not in science alone, after rising above the merely practical, that precision of imagination is useful; in the enjoyment of the fine arts it is equally essential. Not only must the sculptor and the painter draw and model correctly, as a basis for their more delicate touches of expression and feeling,

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but the spectator must have a correct and trained power of imagination in space, or he cannot appreciate and be fed by their work. In music there is an equal necessity for the algebraic imagination of rhythm; but no ordinary mode of teaching algebra cultivates and trains this power, and we therefore omit the consideration of music. In poetry, fiction, and the drama the necessity of a trained and accurate imagination in space is also obvious. These artistic forms of composition all aim to arouse feeling and sentiment, rather than to produce intellectual assent. But feeling is aroused only by the sensible; poetry and fiction, therefore, always employ imagery, or describe what is visible and audible. The reader must, for the descriptive words or sentences, call up in his imagination the scene described or alluded to; and his emotion aroused from the poem will be, in large measure, directly dependent upon the vividness and accuracy with which he portrays the scene. The culture of the geometric imagination, tending to produce precision in the remembrance and invention of visible forms, will, therefore, tend directly to increase the appreciation of works of belles-lettres.

Our attention was first called to this use of geometry, in the culture of the imagination, over thirty years ago, by the late Theodore Strong, of Rutgers College; who, in a private conversation, from which I think that death has taken the seal, said that he thought his own mathematical power lay almost wholly in the vividness and clearness of his imagination of forms in space. We should express the same analysis by saying that his powers were geometrical rather than algebraical. The converse mathematical power lies in this power to conceive of time abstractly in its flow, and then of quantity in a state of continuous expansion or contraction. This is imagination in time, as contrasted with imagination in space, and leads to triumphs in the calculus. But for the highest mathematical success there is a third power requisite, and that is the power of using the language appropriate to the mathematics.

The vocabulary appropriate to discussing questions of time

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and space and of quantity (considered as an abstract relation of time and space, or considered as the finitude of the finite), is somewhat limited, but very flexible, and capable of great precision of expression. The syntax of this language is energetic and forcible, — the symbols by which it is written are marvels of condensation. The most striking characteristic of the written language of algebra and of the higher forms of the calculus is the sharpness of definition, by which we are enabled to reason upon the symbols by the mere laws of verbal logic, discharging our minds entirely of the meaning of the symbol, until we have reached a stage of the process when we desire to interpret our results. The ability to attend to the symbols, and to perform the verbal, visible changes in the position of them permitted by the logical rules of the science, without allowing the mind to be perplexed with the meaning of the symbol until the result is reached which you wish to interpret, is a fundamental part of what is called analytical power. Many students find themselves perplexed by a perpetual attempt to interpret not only the result, but each step of the process. They thus lose much of the benefit of the labor-saving machinery of the calculus; and are, indeed, frequently incapacitated for using it.

A curious example of this incapacity was shown, about thirty years ago, by the successive classes of sophomores in Harvard College being unable to master the reasoning by which Arbogast's Polynomial Theorem was demonstrated. The gist of the reasoning lay in the attempt to prove that Q might be obtained from P. From P we could obtain P, and from Q we might get Q; but P was equal to Q, therefore we could get Q from P, therefore from P. It seems very simple; it is but saying that if we can go from any seaport to the largest commercial port on our Atlantic coast and if we can go from New York to Pittsburg, then we can go from any seaport to Pittsburg — if in no other way, at least, by going by boat to New York, and there taking the railroad. But in this case the ideas of the metropolis and of the modes of travel are clear and familiar, and their

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presence does not confuse the reasoning process. In the algebraical theorem the mind is confused by the attempt to carry, through the reasoning, conceptions of the nature of the four quantities P, P, Q, Q, and of the nature of the processes by which the passage is made from the Roman to the Italic, and back again. It is in vain that you instruct an ordinary student to refrain from the attempt, and simply to confine his attention to the already demonstrated fact, that the passage is easily made and the resulting quantities easily expressed; he will not, or cannot, refrain from entangling himself by making the attempt, and failing.

The language of mathematics, permitting great sharpness and accuracy of definition, conduces largely to their power of drawing necessary conclusions. Language is not only a means of recording the results of our thinking; it is an instrument of thought, and that of the highest value. I do not here refer to the fact that it is impossible for us to convey to another mind our conclusions without the aid of symbols, nor to the fact that we cannot usually bring our inferences to the test of the syllogistic form until we put the process into words; but rather to the broader fact that, until we have linked a conception to the mnemonic of a word or symbol, it is difficult for us to keep it distinctly in our own minds; and also to the equally important consideration, illustrated above, that when we have definitely fixed the meaning of a word or symbol, so that it shall include neither more nor less than we intend, we can discharge our minds of that meaning, and operate upon the symbol as a symbol, by the mere rules of the syntax of the language and the laws of logic, and be confident that every result thus attained upon the symbols as such, will, when properly interpreted, give a truth concerning the things originally symbolized. It is difficult to convey to readers not familiar with the practical use of mathematical symbols any conception of the generality and the minuteness of the conclusions thus frequently attained by a very brief process of transformation, and summed up in a short formula.

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From these sciences of space and time the science of mechanics first borrowed the use of symbolic notation, and learned the art of drawing necessary conclusions. It divides itself into sundry branches. As kinematics, it deals only with the motion of forms, dismissing the idea of force; as statics, it conceives of forces in a state of equilibrium; as dynamics, of forces not balanced, and thus producing motion. Chemistry and logic also followed in this use of symbols; and in manufactures a curious special example was given by Charles Babbage, in the invention of a notation for machinery by which the complex machinery of an engine could be so represented on paper as to enable you to draw necessary conclusions; and the inventor, by the aid of this notation, actually simplified one of his marvellous calculating engines so as to produce by three turns of the handle all the effects which he had before produced by eight — if my memory is right as to the numbers.

Whether the other sciences will ever arrive at a state in which they can use notation and draw necessary conclusions remains to be seen — perhaps by our successors, perhaps by ourselves in the world to come. But as we go upward in the hierarchy of sciences it becomes more difficult to frame definitions, and more difficult to draw necessary conclusions. Shall we therefore say that the reasoning of mathematics is of no use to us in learning to reason upon social, political, moral, and religious questions? By no means. The process of reasoning is similar in all departments of thought, and varies chiefly in the proportion which induction and deduction bear to each other. In geometry the induction is very brief and rapid, while the deductions are sometimes wonderfully long and interdependent. Some of the propositions, absolutely demonstrated, would require two or three hundred connected syllogisms in order to trace them back to self-evident truths. In politics and theology, on the other hand, the induction is slow and cautious, and requiring often the consilience of many inductions to bring us to any tolerable degree of certainty; while the deductions that can be safely

made are very direct and few. Yet in both cases there is inductive and deductive reasoning; and in the mathematics both binds have been carried to the utmost perfection, and may justly serve as models for reasoning in the probable sciences.

When Spinoza undertook to treat theology by the geometrical method he failed, because he took only the deductive part as a model, and did not assume his definitions and axioms with the carefulness and precision of the geometer. A geometrical conception is perfectly clear, and the term applied to it must be absolutely confined to that conception. A circle, for example, is a plane surface bounded by a line everywhere equally distant from a central point in that surface, and in geometry means nothing else — no point out of the plane, and no point in the plane and outside the line, belongs to the circle. Nor can the line be varied, or moved in any manner with reference to the central point, except by taking a new circle, larger or smaller, around the point. The straight line in the plane is a line that runs in every part in the same direction as in any other part. Now, out of these two definitions I frame the propositions, that a given straight line can cut the bounding line of a given circle only in two places; and that, if the line enters the circle, it must, if sufficiently prolonged, emerge from it; in other words, it cannot cut the boundary once without cutting it twice. The reader of these propositions will — if they chance to be new to him, and his interest in them is aroused to give assent or dissent — immediately look at a circle in space, and with the mind's eye look at innumerable straight lines, crossing it in all imaginable ways, simultaneously or consecutively. He will see that they all cross the boundary twice, and he rushes at once to the induction that it will always be so; nay, he says it must be so, because the definition of the circle and the line are so exact, and his conception so clear, that he sees no possible way in which the induction can be invalidated. Many of the inductions of simple molar mechanics are of the like character; they are so rapid, so exhaustive, so free from the

possibility of having overlooked cases, that we call them self-evident, and draw necessary conclusions from them. Not so when we arrive at molecular mechanics; there caution is requisite, and the myriad observations may require years of experimenting. Much less is it so when we reach the highest department — theology. As far as we know anything of theology from the light of nature alone, we gain it by the slowest and most cautious induction. It is true that there are instinctive judgments, and intuitions which give us light by an instantaneous flash. But I am speaking now of a scientific knowledge of theology that may be compared with geometry, when I say that such knowledge is gained only by cautious induction; the intuitions by which the soul recognizes the being of God, may be compared, in regard to geometry, rather to the artistic perception of the beauty of the circle and the ellipse, than to a scientific knowledge of those figures. Spinoza, however, begins with a rapid induction of the idea of substance, so rapid that he may have overlooked many ways in which it could be invalidated; he does not define this idea with absolute precision and definiteness, and yet brings out of it by deduction his long array of theological propositions.

Not such is the manner in which we would apply the lessons of mathematical methods to the reasoning of the statesman and the theologian. The geometer seeks, it is true, for self-evident axioms, and for the means of deducing his theorems from them. Yet, in some cases of great importance, he has been content for a long time to rest upon a careful induction. The binomial theorem of Newton, was the first of many cases in which the law of an infinitely long series was thus obtained from an examination of half a dozen of the first members of the series. There was not the shadow of doubt left concerning the truth of the law thus obtained; and yet it left a desire in the mind of the mathematician to discern a way in which he could found the law on self-evident principles. The student, in the higher branches, must be content, for the most part, with induction; let him learn from the geometer that he must exercise caution and prudence in

forming his induction, but having once with caution and prudence formed a sound judgment, hold to his view with confidence in himself, and in the truth.

One of the most important lessons which mathematical reasoning offers to the theological student is in the treatment of the infinite and of the infinitesimal. These two walls of mystery surround the geometer as they do the theologian. Space is the simplest of all possible objects of human contemplation, so that Hamilton of Edinburgh thought that the only reason that most people are deficient in geometrical power is, that they are too intellectual, and cannot bring their minds down to the contemplation of anything that is so nearly nothing as space. Yet space has its unfathomable mysteries in the infinite, and in the infinitesimal. To illustrate by an easy example in the infinitesimal, consider the simple idea of a curved line,— what is it? A line that bends continually and continuously. But these attributes are contradictories. If the line bends continually, then it bends at a given point in it, A, and changes its direction at A. There is, then, an angle or corner, however blunt, at A. But it bends continuously; there are no angles, and therefore none at A. The geometer is obliged to recognize this contradiction, and in his reasoning assumes that the line bends at A, or does not bend — either, neither, or both; and from any one of the four assumptions can deduce correct, or by carelessness erroneous, conclusions.

An easy example of the mystery in the infinite may be formed by imagining two lines rotating in a plane about their point of intersection. To render it easier to represent, let one line lie stationary, pointing north and south, while the other rotates. Now, when the rotating line lies east and west, let it extend one foot from the centre, but let it grow longer, as it approaches the north and south line, in exact proportion to its approach. That is to say, when pointing east its length is one foot; when north-east, two feet; when north-north-east, four feet; when north-by-east, eight feet, and so on; when it has swung past the meridian to north-by-

west, let it be again eight feet, at north-north-west four feet, and so on. It will at once be seen that on the meridian the line will be of infinite length, and that we cannot say at what point the end will cross the meridian. But that it does cross it is evident, because we swing the line from pointing easterly to pointing westerly. Yet it cannot cross, for the end constantly recedes from the meridian, as the line swings nearer to it — when pointing east the end is one foot from the meridian, and when we have swung it so near to the meridian that Arabic figures would fail to express its length, the end would be more than eighteen inches off, and the distance never could become less. Here, again, the geometer acknowledges that the results are contradictory, but, nevertheless, knows that both are true. He has proved conclusively that the law of non-contradiction does not hold for infinites or infinitesimals. Would that the theologian and metaphysician might learn it also. Here is the simplest possible species of infinite — the infinite of length — coming in to confuse irreconcilably two ideas which would seem also to be the simplest and clearest of all conceptions — the conception of a straight line rotating in one plane about a point in the line itself. It seems that nothing can prevent our imagining that it swings into and past the meridian drawn through that point. It seems that no increase in the length of the line can do more than make its end cross the meridian at an infinite distance. Yet, if it increases with sufficient rapidity, the end may recede from the meridian not only slowly, as in the case just given, but rapidly, and recede forever. If, for example, instead of doubling, we had quadrupled, the length of the line every time that we halved the angle, the end would have receded to an infinite distance east of the meridian. But, keeping to our original example, there is no difficulty in imagining the line to rotate until it coincides with the meridian; there is no difficulty in imagining the line to increase in length in the inverse ratio of its angle with the meridian; and there is no difficulty in demonstrating that its end will always recede from the meridian, and approach

a parallel line 1.5708 feet to the east of it. Yet this last demonstration is irreconcilably at variance with the first conception. If the introduction of the infinite thus introduces insoluble contradiction among these clearest and simplest possible of human conceptions, how well does it become us to hesitate in regard to higher matters of the infinite God and his attributes; to hesitate about accepting or rejecting either logical deductions, or the results of inductions, or apparent intuitions, when they involve infinity among their elements.

The Darwinian hypothesis that the slow variation of animals from their ancestral types has, through the variable circumstances around the animal, favoring certain variations in one place and not in another, produced all the variety of animals upon earth, is an example of the very loose and unjustifiable way in which infinity is sometimes introduced even into the physical sciences. It is acknowledged that no instance is known of the accidental variation of a species producing any approximation toward another species, except in the most external and least important particulars; but it is assumed that an infinite time would do what a finite time shows no tendency to do.

It is not thus that the geometer deals with the infinite and the infinitesimal. He obtains them by an irresistible induction from the finite, and never ventures to use them unless thus obtained. When his quantities become unmanageable from their magnitude, or from their minuteness, he ceases to use them; and uses only the ratio between them, which he has discovered by reasoning upon them in their finite magnitudes. In other words, the geometer draws from the consideration of finites certain conclusions concerning the finite relations of infinites and infinitesimals. From those relations he deduces again finite conclusions. But he never introduces the infinites and infinitesimals themselves as premises into his reasoning; nor is he able to discover their relations to each other, unless he has finite premises to start from. Otherwise he declares those relations to be absolutely indeterminate.

This is the Darwinian condition — that theory assumes infinitesimal variations and infinite times, and draws thence finite conclusions, without the slightest warrant from reason for the large drafts on the imagination.

A celebrated lawyer of Massachusetts was once much berated for speaking of the glittering generalities of the American Declaration of Independence. We shall not attempt his defence; for we would not join in anything like a disrespectful treatment of that immortal document. Yet the higher mathematics give us many proofs of the truth which lay concealed under his reproachful epithet, and which gives the basis for the popular distinction between theory and practice. It is painful to see how many men, otherwise sensible, retain a strong prejudice against books and against scientific men. After Agassiz has restocked the rivers of both Europe and America with valuable fish, and thus added to the annual income of practical men hundreds of thousand of dollars, we find men still sneering at pisci-culture, as mere scientific theory, and not a thing of practical utility. The causes for this prejudice against scientific theory are twofold. In the first place there are as many pretenders and charlatans in science as there are quacks in medicine, or weak heresiarchs in religion. An agricultural editor and lecturer who tells his audience that a cat runs up a board fence by means of vacuum suckers in her feet, should not expect much deference to his opinions upon any subject. But, in the second place, there is much truth which feeds the intellect and the heart which, nevertheless, we do not as yet know how to apply to practical uses. It is not for that reason worthless. Knowledge is an end in itself; the body is the servant of the soul, not the soul of the body; and I need food and clothing in order that I may study geometry, and teach geometry; I do not study and teach that divine science for the sake of obtaining food and clothing. Nor can we foresee what uses may hereafter come, of truths which now seem barren of applications. Many of the fine discoveries of the Greek geometers lay idle for fifteen or eighteen centuries before their value was rec-

ognized by other thinkers. “Art is long,” and gathers her tools from among the stores of all ages.

Many of the finest inventions of our own and other times have been, as it were, accidental discoveries by unlearned men; thus much must be conceded to the practical objector to science. But, on the other hand, a majority of the most useful practical inventions of the present century have been the direct result of scientific labor, undertaken for scientific ends alone, without a thought of pecuniary benefit or commercial use; and it would be a suicidal policy for the men of business to discourage the pursuit of the highest and most abstract science on the ground of its want of practical utility. I have just alluded to the artificial impregnation of fish eggs; I might allude to the discovery of trichina; to the aniline dyes; to electrotyping, electroplating, and the magnetic telegraph, and to twenty other notable things of the present day — the gifts of pure science to the practical world.

Nevertheless, the concession must also be made that many of the most beautiful theorems of science delight us by their beauty and truthfulness alone, without regard to ulterior benefit. They minister to our highest needs; they satisfy the noblest ambition of the intellect, to discover truth; the purest sentiment, to admire beauty; but they do not show directly any mode by which we can produce a merchantable commodity. In the higher mathematics we have a curious illustration of this fact, since many of the finest theorems are not only incapable of being applied to practical problems, but are incapable of being used for the advancement even of mathematical problems. Some special problem has been started and solved; the solution immediately led the mathematician to a theorem of great generality, including the solution of innumerable problems; and yet this theorem, demonstrably true, and including the solution of all these problems, cannot lead you to the actual solution of a single one of them; that is to say, each one of them if solved at all must be attacked in some special way; or else be solved only by a method of approximation leading toward, but not actually to, the correct result.

To me this seems analogous to the fact that the widest moral axioms and loftiest moral sentiments are often found to be inapplicable, by our weak understandings, to a practical question of social or political duty; and we are obliged to make very coarse approximation to it. And although John Locke esteemed it a weakness of the human understanding that it would so often fain comfort itself in its conclusions by analogy, I would confess my human weakness, and acknowledge that I have often derived great comfort from having my faith in the eternal verities of ethical science thus confirmed by finding them subject to no more limitations of practicability than the eternal theorems of geometry.

In the year 1692, James Bernouilli, discussing the logarithmic spiral (that is, the spiral which makes at every part of it the same angle with a line drawn to its pole), shows that it reproduces itself in its evolutes, its involutes, and its caustics of both reflection and refraction, and then adds: “But since this marvellous spiral, by such a singular and wonderful peculiarity, pleases me so much that I can scarce be satisfied with thinking about it, I have thought that it might be not inelegantly used for a symbolic representation of various matters. For since it always produces a spiral similar to itself, indeed precisely the same spiral, however it may be involved or evolved, or reflected or refracted, it may be taken as an emblem of a progeny always in all things like the parent, *simillima filia matri.* Or, if it is not forbidden to compare a theorem of eternal truth to the mysteries of our faith, it may be taken as an emblem of the eternal generation of the Son, who as an image of the Father, emanating from him, as light of light, remains ὁμοούσιος with him, howsoever overshadowed. Or, if you prefer, since our *spira mirabilis* remains, amid all changes, most persistently itself, and exactly the same as ever, it may be used as a symbol, either of fortitude and constancy in adversity, or, of the human body; which after all its changes, even after death, will be restored to its exact and perfect self; so that, indeed, if the fashion of imitating Archimedes were allowed in these days, I should

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gladly have my tombstone bear this spiral, with the motto, Though changed, I rise again exactly the same, Eadem numero mutata resurgo.”

The disposition which Bernouilli thus shows, to use a geometrical figure in illustration of moral truth, is inherent in human nature. Inasmuch as all our language is necessarily drawn in the first instance from that which is external to ourselves, and to which we can point in illustration of our meaning, it is natural that we should use freely geometrical form or figure, the first property of outward things which becomes the distinct object of intellectual apprehension. Thus we get in our familiar speech concerning moral and spiritual themes, such mathematical words as upright, straightforward, square dealing, rectitude, obliquity, tortuous, crooked, diametrically opposite, and many others. That which men in general thus freely do, without question as to its appositeness and propriety, the mathematician may surely be allowed to do to a larger extent. The more extensive a persons acquaintance with any subject, the more frequently will topics selected from it arise to his memory, by Locke’s weakness, to illustrate other departments of thought. A special instance is given in the foregoing extract from Bernouilli; and other instances will come readily to the remembrance of those familiar with the history of mathematics and mathematicians. In one of the Bridgewater treatises some remark was made to the effect that the study of that rigid science rather hindered the mind from the appreciation of religious thought. Charles Babbage, one of the successors of Isaac Barrow and Isaac Newton in the Lucasian chair at Cambridge, replied to the taunt by writing a ninth Bridgewater treatise, in which he illustrated the subject both of revealed and natural religion, with wonderful power and beauty, by arguments and analogies drawn from the most recondite researches of the mathematician.

One of the most remarkable of Babbage's illustrations of miracles has never had the consideration in the popular mind which it deserves; the illustration drawn from the existence

of isolated points fulfilling the equation of a curve. The mathematicians definition of a curve requires more than the existence of a line bending everywhere imperceptibly, and thus having in it neither any angle nor any straight portion; that is an artist's description. The mathematician's definition requires that the position of every point in the line should be described by describing the position of one point; as, for example, to say that a certain point is at a given distance from a given point, describes the position of every point in the circumference of a circle with the given point as a centre, and the given distance as radius; but of no other point whatever. There are, however, definitions of curves which will describe not only the position of every point in a certain curve, but also of one or more perfectly isolated points; and if we should attempt to get by induction the definition, from the observation of the points in the curve, we might fail altogether to include these isolated points; which, nevertheless, although standing alone, as miracles to the observer of the course of the points in the curve, are nevertheless rigorously included in the law of the curve. This illustration not only proves the possibility of the scripture miracles being in accordance with the highest laws of nature, and expressly provided for in the original constitution of matter — for which Babbage uses it; but it has a direct bearing also on the great question which is so deeply stirring the world to-day — the question whether the doctrine of evolution is to be accepted or rejected.

On *a priori* grounds we should expect to find that the ordinary generation by descent had been going on unbroken from the beginning of life on the planet; that existing animals and plants were the offspring of the extinct races in the preceding age; and they the offspring of those of a still more remote age, and so on to the beginning. Many naturalists think that observed facts also warrant the induction that this hypothesis is true. Charles Darwin follows it out with a prodigious wealth of learning, endeavoring to show that he has discovered the causes which have changed the extinct

into the living species; he finds them in two modifiers of the general law that like produces like; the first being a slight disposition in the offspring to vary a little from the likeness to the parent; the second being in the adaptation of these variations, under some circumstances to favor, under others to injure, the chance for life. But there are several fatal objections to Darwin's view, which, in spite of the popularity given to it by the ingenious and plausible arguments of its advocates, will finally cause it to be laid aside. One of these consists in the fact that there is in the rocks no record of any continuous change; the variation is always abrupt and by discrete steps; and the Darwinian explanation by blanks in the record, fails for two reasons: first, the inherent incredibility of the blanks being always so immensely longer than the records; secondly, the great probability, from the temperature of the earth and sun, that the records themselves fill out all the time that has elapsed since the earth was red-hot at the surface. The evolutionist, however, not a Darwinian, may appeal to the existence of singular points in a curve, and say, The law of descent is not fully understood by us, and it has been subject to abrupt variations in its effects, without any variation in the real law itself. Another illustration of the same point might be drawn from the abrupt transition frequently made in the character of a curve by a change in some of the so-called constants in its law. If you change the diameter of a circle, you change only the size, not the shape of the curve. But if you change the proportion between the principal diameters of an ellipse, you obtain circles, ellipses, parabolas, hyperbolas, and even straight lines, parallel, or at right angles. Now it may be said that in the law of hereditary descent, there is a gradual change of the constants, betraying itself at certain points by a sudden change in the character of the offspring.

Agassiz was an earnest opponent not only of Darwin's views, but of every form of the doctrine of evolution of one species from another. He appeared to regard the discerning of differences as a higher and more important work than the

discovery of likenesses. The evolutionists appear always to confound likeness with identity. A few years since it was the fashion for botanists and zoologists to multiply divisions, and divide old groups into new; of late years the inclination has been strong to deny the real meaning of any divisions. The number of divisions, it is said, depends altogether on the number of specimens you have; given specimens enough, and your divisions will disappear because you can find no place to make them. Each species will vary toward the cognate species until you always find individuals that can be referred to one species as easily as to the other. In Whewell's Philosophy of the Classificatory Science, he defines the members of a group to be those individuals which, on the whole, bear a stronger resemblance to the type of the group, than to any other type. He thus allows for that free variation which may produce these ambiguous members. His philosophy does not, however, suit those of our day who have adopted Darwin's views, and they press the fact of a frequent insensible gradation in individuals from one form to another as a sufficient proof of the identity of the two forms. To me the argument never seemed sufficient. I was talking one day on the subject with a botanist of great eminence, and he adduced the instance of the oak and the chestnut. "You cannot," he said, "frame a definition of oaks that will not cover some chestnuts, nor one of chestnuts that will not cover some oaks." "But," said I, "can you imagine a plant which you could not refer to one more readily than to the other?" "No," said he, "I confess that I think I could always discriminate."

Many of our clearest distinctions are thus made from indications which are certain and unmistakeable to the expert, although they are not capable of being so expressed in a formula of words as to enable the less expert to detect them. My botanical friend would argue the identity of oaks and chestnuts from their shading together in the chinquapin forms; yet, on being pressed, confessed his secret conviction that he could tell one from the other under any disguise. And I

confess my conviction that in all cases in which allied species, or allied genera, run toward what are called connecting links, the difference is as real in the connecting links, as in the typical members; as real, but less conspicuous.

The case can be illustrated by geometrical curves in this way. Take any two curves of the most strikingly different forms, and of an entirely different genesis; you may, by a change of the constants, bring them together to an intermediate form; but you cannot by any possibility convert one into the other. The intermediate form, although geometrically (that is in shape) the same, is yet intellectually, in the law of its formation, different, according to your mode of approach; and when you have approached it in one direction you cannot recede in another, without an intellectual *saltum,* a forsaking one conception and taking up the other. Let the chestnut be represented by a long ellipse, for example, and by changing the ratio of the diameters we can bring it to the chinquapin form of a circle. Let the oak be represented by the *spira mirabilis,* and, by making the angle with the meridian become a right angle, we can bring that also to its chinquapin form of a circle. But we have not thereby rendered it possible to consider the spiral and the ellipse different forms of the same curve, or possible to pass by any law of variation from one to the other. The circle obtained from the ellipse, although of precisely the same size and form as that obtained from the spiral, is nevertheless intellectually and algebraically a totally different curve; and although you can return from it to the ellipse, you cannot pass from it to Bernoulli's curve. That which makes an oak an oak is no mere outward form and appearance, but some intellectual law embodied in the ovule and in the pollen, and unfolding itself in the tree; and it cannot unfold into the chestnut unless it be changed in its very essence. This is a case in which the P is not really equal to Q. Neither let any man say that this analogy of the curve is far-fetched. The trees are forms in space, undergoing periodic changes; the law which builds up the atoms into these forms, must be a law of space and time.

There are many special examples in the mathematics which may be used for the illustration of special points in morals or politics. In the first volume, for example, of Peirce's Curves and Functions, there is a means provided of cutting a curve abruptly off at any point, by adding a term which shall be of infinitesimal weight, and produce no effect on the part of the curve which you wish to preserve, but the moment you attempt to pass the boundary shall become imaginary and reduce the whole to absurdity. It always seemed to me a striking symbol of certain mischief-makers, who seem powerless to do any good in the world; they are mere ciphers in doing anything useful, and in discovering truth; but prove effective hinderances in the way of others accomplishing good. Or it may be a symbol of the paralysing power of doubt. No man can accomplish anything worth doing, unless he acts in faith, with a lively and earnest zeal, born of conviction. Some doubter meets him, and suggests objections and criticism adverse to his conclusions. It may not destroy his conviction altogether, but it palsies his faith and zeal, and destroys his energy and usefulness.

There is a certain spiral of a peculiar form on which a point may have been approaching for centuries the centre, and have nearly reached it, before we discover that its rate of approach is being accelerated. The first thought of the observer, on seeing the acceleration-, would be to say that it would reach the centre sooner than he had before supposed. But as the point draws near the centre it suddenly, although still moving under the same simple law as from the beginning, makes a very short turn upon its path and flies off rapidly almost in a straight line, out to an infinite distance. This illustrates that apparent breach of continuity which we sometimes find in a natural law; that apparently sudden change of character which we sometimes see in man.

The most important effect, perhaps, upon the mind of a true student of mathematics, is the habit which he gains of recognizing the presence of thought in the material universe. The things which are seen are temporal, but those which are

not seen are eternal. The geometer contemplates the realities of space and time until they become to him more real than the passing things of the world. To him, first of all men, the physical manifestations of the universe all became mere "modes of motion." He saw that the atoms of matter as manifesting force, can manifest it only in space and time, and the only phenomenon conjoining space and time is motion. Thus the ultimate analysis of a sensible property must result in giving motion as the cause, and if the atoms move according to any law, it must be an algebraic and geometric law. Henceforth, the investigation of these laws, as mathematical conceptions, became his chosen labor. Thus he was freed at once from the limitations of magnitude; the grandest starry system moves by the same laws that govern the Jovian satellites; and those laws would possess the same intellectual interest and grandeur if ruling the motes under the microscope. The whole interest of the physical universe, in its merely physical manifestations, concentres, therefore, to the mathematician in the laws of geometry and algebra which it embodies. The creation is but a set of diagrams wherewith the Infinite Teacher would illustrate for us mathematical thoughts and lead us to their perception, and thus to a reverent recognition of his existence and of our likeness to him.

The students of the other physical sciences are gradually beginning to perceive that they need, and must have, the aid of the mathematician in their work. Molar mechanics first yielded, primarily in its grandest form of astronomy, afterwards in regard to terrestrial phenomena; then molecular mechanics, in optics, electricity, and other branches, began also to yield, not without some conflict. The sciences of natural history have yielded but little; and I have recently heard some of their highest scholars talking almost as bitterly against the dogmatism and conceit of the mathematician, as ever others have done against the tyranny of theology. The resistance is vain. There is but one universe in which we live, and it is a cosmos, pervaded with one spirit of beauty

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and truth. The hierarchy of sciences which would interpret this cosmos must needs begin with space and time, and rise to God; and all the intermediate sciences of every kind must own the absolute sovereignty of those at each end of the series; there is no break, — what is physical is subject to the laws of mathematics, and what is spiritual to the laws of God, and the laws of mathematics are but the expression of the thoughts of God. Botany and zoology, in their classification, are in the condition in which crystallography was before there was any other classification of crystals than by the figures of the ancient geometry; and as the needs of the mineralogist called forth from the mathematician a new geometry by which the science assumed finer intellectual proportions, so the needs of the botanist and zoologist will, when the epidemic of “natural selection “has spent its fury, call out new contributions from the mathematicians in the line in which Peirce began, when, in 1849, he showed that the plants embody the idea of extreme and mean ratio, and in which Chauncy Wright followed, in showing that this ratio distributes the leaves most evenly about the stem. It will finally be shown that the various forms of plants and animals are the embodiment, in their forms, of distinct mathematical ideas; that Agassiz is right in his Essay upon Classification, in declaring the presence of a connected plan, an intellectual scheme, in each of the great kingdoms of organic nature. And if we can clearly bring those mathematical ideas into algebraical form, we can decide upon the possibility of the evolution of one species into another, or upon the impossibility. To the mathematician the study of organic forms has always appeared as the most glorious exercise in which the geometric intellect could be engaged; and the truth of Darwin’s theory of a stumbling, staggering, hap-hazard production of these forms, would seem to the mathematician like the destruction of his sublimest hope of happiness in this life or in the life to come. Of course, if inexorable facts sustain Darwin, the mathematician must submit to this disappointment; but for the present he must be excused for his incredulity when he is

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told that the organic world, which seems to him a volume of the richest geometrical and algebraical learning, full of the most beautiful geometrical forms, and the most wonderful algebraic laws of periodicity, — the meaning of which he has in many instances clearly unfolded to himself, — has, in reality, no meaning whatever, but is like pages set from a jumble of types, not even set in lines by the hand of a printer, but which have been accidentally sorted by falling over sieves, and accidentally brought into pages by being shaken accidentally in shallow boxes of the size of a page; and illustrated by wood cuts drawn and engraved by the accidental scratching of a block, kicked along a gravel walk. If the mathematician is ever brought to admit the theory of evolution, he will still hold it demonstrated that the evolution has proceeded according to intellectual laws, evolving simply what was contained in the original idea of the creation. Before the world existed in its actuality it existed in idea, and so far as that idea related to the external world, it was a mathematical idea, — an idea of form and period, that is of motion. To this spiritual view of the origin of the universe the mathematician must cling, whatever his views concerning evolution. He discovers everywhere in nature a manifest embodiment of his own *a priori* ideas of space, time, number, and mechanism. And in the words of the geometer at our Cambridge: “The solution of the problem of this universal presence of such a spiritual element is obvious and necessary. There is one God, and science is the knowledge of him.”
